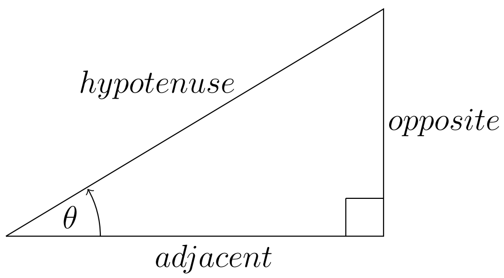
Trigonometric Formula Sheet

Definition of the Trig Functions

Right Triangle Definition

Assume that:

$0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$



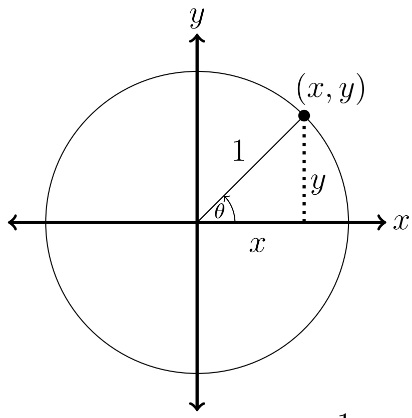
$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

Unit Circle Definition

Assume θ can be any angle.



$$\sin\theta = \frac{y}{1} \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Domains of the Trig Functions

$\sin\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\cos\theta, \quad \forall\ \theta \in (-\infty, \infty)$

$\tan\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\csc\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

$\sec\theta, \quad \forall\ \theta \neq \left(n + \frac{1}{2}\right)\pi,\ where\ n \in \mathbb{Z}$

$\cot\theta, \quad \forall\ \theta \neq n\pi,\ where\ n \in \mathbb{Z}$

Ranges of the Trig Functions

$-1 \leq \sin\theta \leq 1$

$-1 \leq \cos\theta \leq 1$

$-\infty \leq \tan\theta \leq \infty$

$\csc\theta \geq 1\ and\ \csc\theta \leq -1$

$\sec\theta \geq 1\ and\ \sec\theta \leq -1$

$-\infty \leq \cot\theta \leq \infty$

Periods of the Trig Functions

The period of a function is the number, T, such that f (θ +T) = f (θ) .
So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \csc(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega} \qquad \sec(\omega\theta) \ \Rightarrow\ T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega} \qquad \cot(\omega\theta) \ \Rightarrow\ T = \frac{\pi}{\omega}$$

Identities and Formulas

Tangent and Cotangent Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$ $\quad$ $\csc\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{1}{\sec\theta}$ $\quad$ $\sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{1}{\cot\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Even and Odd Formulas

$\sin(-\theta) = -\sin\theta$ $\quad$ $\csc(-\theta) = -\csc\theta$

$\cos(-\theta) = \cos\theta$ $\quad$ $\sec(-\theta) = \sec\theta$

$\tan(-\theta) = -\tan\theta$ $\quad$ $\cot(-\theta) = -\cot\theta$

Periodic Formulas

If n is an integer

$\sin(\theta + 2\pi n) = \sin\theta$ $\quad$ $\csc(\theta + 2\pi n) = \csc\theta$

$\cos(\theta + 2\pi n) = \cos\theta$ $\quad$ $\sec(\theta + 2\pi n) = \sec\theta$

$\tan(\theta + \pi n) = \tan\theta$ $\quad$ $\cot(\theta + \pi n) = \cot\theta$

Double Angle Formulas

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then:

$$\frac{\pi}{180^\circ} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180^\circ} \quad \textbf{and} \quad x = \frac{180^\circ t}{\pi}$$

Half Angle Formulas

$\sin\theta = \pm\sqrt{\dfrac{1-\cos(2\theta)}{2}}$

$\cos\theta = \pm\sqrt{\dfrac{1+\cos(2\theta)}{2}}$

$\tan\theta = \pm\sqrt{\dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}}$

Sum and Difference Formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

Product to Sum Formulas

$\sin\alpha\sin\beta = \dfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = \dfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \dfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

Sum to Product Formulas

$\sin\alpha + \sin\beta = 2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

$\sin\alpha - \sin\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

$\cos\alpha + \cos\beta = 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

$\cos\alpha - \cos\beta = -2\sin\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

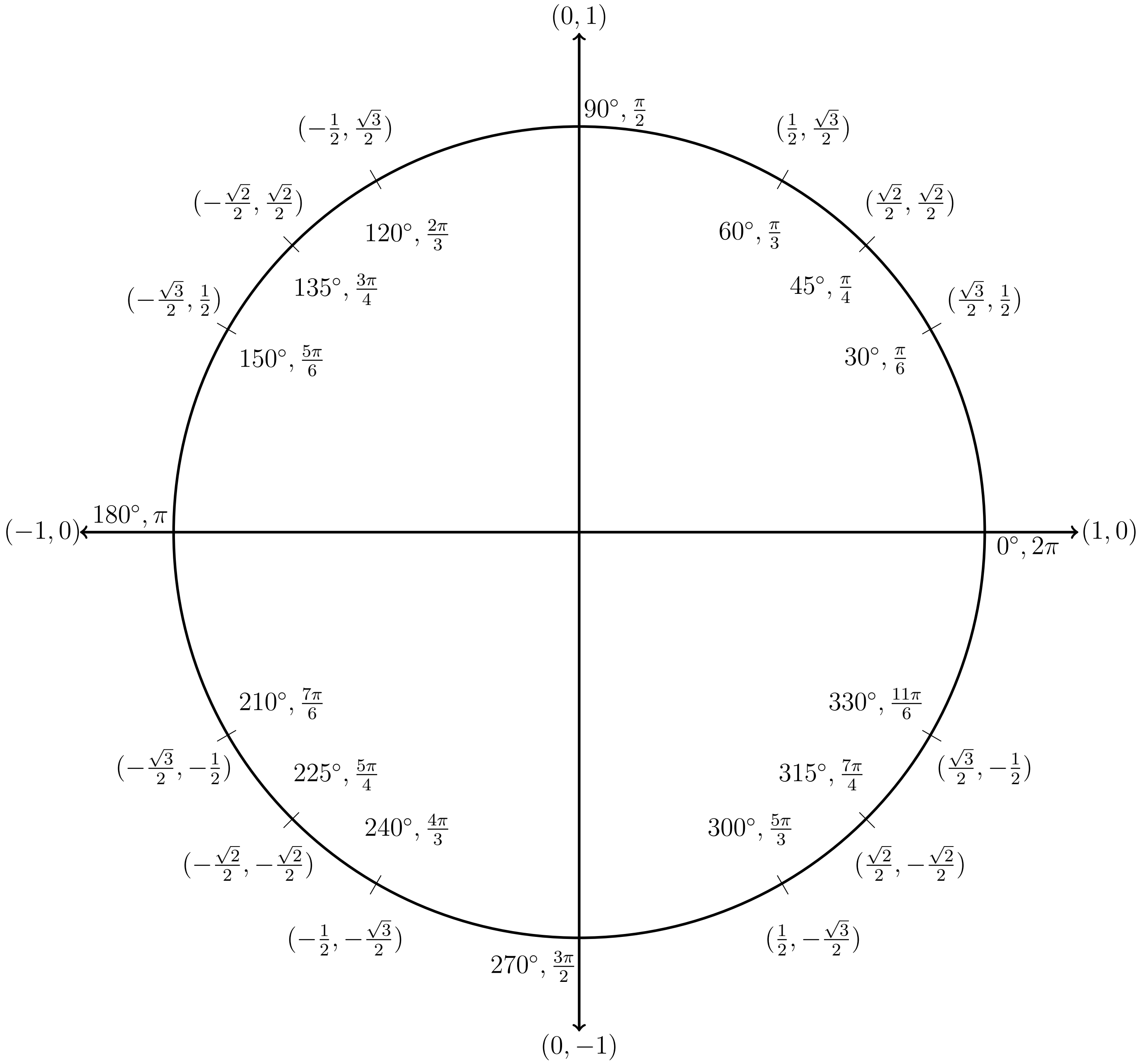
Cofunction Formulas

$\sin\left(\dfrac{\pi}{2}-\theta\right) = \cos\theta$ $\quad$ $\cos\left(\dfrac{\pi}{2}-\theta\right) = \sin\theta$

$\csc\left(\dfrac{\pi}{2}-\theta\right) = \sec\theta$ $\quad$ $\sec\left(\dfrac{\pi}{2}-\theta\right) = \csc\theta$

$\tan\left(\dfrac{\pi}{2}-\theta\right) = \cot\theta$ $\quad$ $\cot\left(\dfrac{\pi}{2}-\theta\right) = \tan\theta$

Unit Circle



For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ *and* $\sin\theta = y$

Example

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \qquad \sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Inverse Trig Functions

Definition

$\theta = \sin^{-1}(x)$ **is equivalent to** $x = \sin\theta$

$\theta = \cos^{-1}(x)$ **is equivalent to** $x = \cos\theta$

$\theta = \tan^{-1}(x)$ **is equivalent to** $x = \tan\theta$

Inverse Properties

These properties hold for x in the domain and θ in the range

$\sin(\sin^{-1}(x)) = x$	$\sin^{-1}(\sin(\theta)) = \theta$
$\cos(\cos^{-1}(x)) = x$	$\cos^{-1}(\cos(\theta)) = \theta$
$\tan(\tan^{-1}(x)) = x$	$\tan^{-1}(\tan(\theta)) = \theta$

Domain and Range

Function	**Domain**	**Range**
$\theta = \sin^{-1}(x)$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\theta = \cos^{-1}(x)$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\theta = \tan^{-1}(x)$	$-\infty \le x \le \infty$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

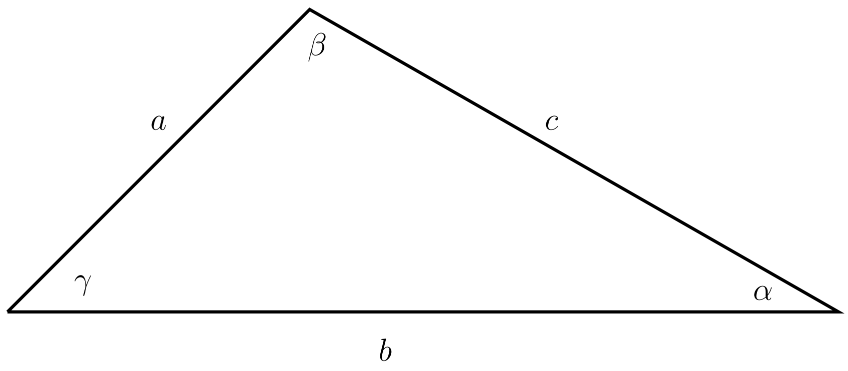
Other Notations

$\sin^{-1}(x) = \arcsin(x)$

$\cos^{-1}(x) = \arccos(x)$

$\tan^{-1}(x) = \arctan(x)$

Law of Sines, Cosines, and Tangents



Law of Sines

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha-\beta)}{\tan\frac{1}{2}(\alpha+\beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

Complex Numbers

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1$$

$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$

$(a+bi)+(c+di) = a+c+(b+d)i$

$(a+bi)-(c+di) = a-c+(b-d)i$

$(a+bi)(c+di) = ac-bd+(ad+bc)i$

$(a+bi)(a-bi) = a^2+b^2$

$|a+bi| = \sqrt{a^2+b^2}$ **Complex Modulus**

$\overline{(a+bi)} = a-bi$ **Complex Conjugate**

$\overline{(a+bi)}(a+bi) = |a+bi|^2$

DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$, and let n be a positive integer.
Then:

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Example: Let $z = 1-i$, find z^6.

Solution: First write z in polar form.

$$r = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$$

$$\theta = arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

Polar Form: $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

Applying DeMoivre's Theorem gives :

$$\begin{aligned} z^6 &= \left(\sqrt{2}\right)^6\left(\cos\left(6\cdot -\frac{\pi}{4}\right) + i\sin\left(6\cdot -\frac{\pi}{4}\right)\right) \\ &= 2^3\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right) \\ &= 8(0+i(1)) \\ &= 8i \end{aligned}$$

Finding the nth roots of a number using DeMoivre's Theorem

Example: Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$.

We are asked to find all complex fourth roots of 4.
These are all the solutions (including the complex values) of the equation $x^4 = 4$.

For any positive integer n , a nonzero complex number z has exactly n distinct nth roots.
More specifically, if z is written in the trigonometric form $r(\cos\theta + i\sin\theta)$, the nth roots of z are given by the following formula.

$$(*)\ r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{360^\circ k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360^\circ k}{n}\right)\right), \quad for \quad k = 0, 1, 2, ..., n-1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r = \sqrt{(a)^2 + (b)^2} \qquad \text{and} \qquad \theta = arg(z) = \tan^{-1}\left(\frac{b}{a}\right).$$

So we have the complex number $a + ib = 4 + i0$.

Therefore $a = 4$ and $b = 0$

So $r = \sqrt{(4)^2 + (0)^2} = 4$ and
$\theta = arg(z) = \tan^{-1}\left(\frac{0}{4}\right) = 0$
Finally our trigonometric form is $4 = 4(\cos 0^\circ + i\sin 0^\circ)$

Using the formula $(*)$ above with $n = 4$, we can find the fourth roots of $4(\cos 0^\circ + i\sin 0^\circ)$

- For $k = 0$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4} + \frac{360^\circ * 0}{4}\right) + i\sin\left(\frac{0^\circ}{4} + \frac{360^\circ * 0}{4}\right)\right) = \sqrt{2}\left(\cos(0^\circ) + i\sin(0^\circ)\right) = \sqrt{2}$
- For $k = 1$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4} + \frac{360^\circ * 1}{4}\right) + i\sin\left(\frac{0^\circ}{4} + \frac{360^\circ * 1}{4}\right)\right) = \sqrt{2}\left(\cos(90^\circ) + i\sin(90^\circ)\right) = \sqrt{2}i$
- For $k = 2$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4} + \frac{360^\circ * 2}{4}\right) + i\sin\left(\frac{0^\circ}{4} + \frac{360^\circ * 2}{4}\right)\right) = \sqrt{2}\left(\cos(180^\circ) + i\sin(180^\circ)\right) = -\sqrt{2}$
- For $k = 3$, $4^{\frac{1}{4}}\left(\cos\left(\frac{0^\circ}{4} + \frac{360^\circ * 3}{4}\right) + i\sin\left(\frac{0^\circ}{4} + \frac{360^\circ * 3}{4}\right)\right) = \sqrt{2}\left(\cos(270^\circ) + i\sin(270^\circ)\right) = -\sqrt{2}i$

Thus all of the complex roots of $x^4 = 4$ are:

$\mathbf{\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i}$.